Int J Theor Phys (2007) 46: 2326–2338
DOI 10.1007/s10773-007-9351-0

Connection between Group Based Quantum Tomography and Wavelet Transform in Banach Spaces

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Published online: 25 April 2007

Abstract The intimate connection between the Banach space wavelet reconstruction method for each unitary representation of a given group and some of well known quantum tomographies, such as: tomography of rotation group, Spinor tomography and tomography of Unitary group, is established. Also both the atomic decomposition and Banach frame nature of these quantum tomographic examples is revealed in details.

Keywords Quantum tomography · Wavelets · Banach space · Group representations · Frames

1 Introduction

The mathematical theory of wavelet Transform finds nowadays an enormous success in various fields of science and technology, including treatment of large databases, data and image compression, signal processing, telecommunication and many other applications [1]. After the empirical discovery by Morlet [2], it was recognized from the very beginning by Grossmann, Morlet, Paul and Daubechies [3] that wavelets are simply coherent states associated to affine group of the line (dilations and translations) [4, 5]. Thus, immediately the stage was set for a far reaching generalization [3, 6]. Unlike function which form orthogonal bases for space, Morlet wavelets are not orthogonal and form frames. Frames are the set

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of functions which are not necessarily orthogonal and which are not linearly independent. Actually, frames are a repeatable set of vectors in Hilbert space which produces each vectors in space with a natural representation.

Recently another concept called atomic decomposition have played a key role in further mathematical development of wavelet theory. Indeed atomic decomposition for any space of function or distribution aims at representing any element in the form of a set of simple function which are called atoms [7]. As far as the Banach space is concerned, Feichtinger–Grocheing [9] provided a general and very flexible way to construct coherent atomic decompositions and Banach frames for Banach spaces.

The concept of a quantum state represents one of the most fundamental pillars of the paradigm of quantum theory. Usually the quantum state is described either by state vector in Hilbert space, or density operator or a phase space probability density distribution (quasi-distributions). The quantum states can be determined completely from the appropriated experimentally data by using the well known technique of quantum tomography or better to say tomographic transformation.

A general framework is already presented for the unification of the Hilbert space wavelets transformation on the one hand, and quasi-distributions and tomographic transformation associated with a given pure quantum states on the other hand [8]. Here in this manuscript we are trying to establish the intimate connection between the Banach space wavelet reconstruction method developed by Feichtinger–Grocheing [9, 10] and some of well known quantum tomographies associated with mixed states, such as: tomography of rotation group [11–14], Spinor tomography [15, 16], discrete spin tomography [17] and tomography of Unitary group [18, 19], all which can be represented by density matrices. Since the density matrix can be presented through Banach space in quantum physics [20]. Therefore, it is natural to do quantum tomography of any density matrix by using the wavelet transform and its inverse in Banach space connected with the corresponding group representation associated with that density matrix. The quantum tomography used by this method for the mixed quantum states is completely consistent with other commonly used methods. Also both the atomic decomposition and Banach frame nature of these quantum tomographic examples is revealed in details.

The paper is organized as follows:

In Sect. 2 we define wavelet transform and its inverse for each unitary representation of a given group in Banach space and then define atomic decomposition and Banach frame in Banach space. In Sect. 3 we first obtain the quantum tomography associated with the unitary representation of a given group in Banach space and then define its atomic decomposition and Banach frame bounds. The section is ended with the derivation of some typical quantum tomographic examples, such as: tomography of rotation groups, Spinor tomography and tomography of unitary group, by using the Banach space wavelet reconstruction method. The paper is ended with a brief conclusion.

2 Wavelet Transform, Frame, and Atomic Decomposition on Banach Spaces

The following is a brief recapitulation of some aspects of the theory of wavelets, atomic decomposition, and Banach frame. We only mention those concepts that will be needed in the sequel, a more detailed treatment may be found in, for example, [7, 9, 21, 22].

Let G be locally compact group with left Haar measure $d\mu$ and let π be a continuous representation of a group G in a (complex) Banach space $\mathcal{B}$.

A representation for group $G \times G$ in the space $\mathcal{L}(B)$ of bounded linear operators acting on Banach space $\mathcal{B}$: is defined as:

$$T : G \times G \to \mathcal{L}(\mathcal{L}(\mathcal{B})) : \hat{O} \to U(g_1^{-1})\hat{O}U(g_2), \tag{2.1}$$

where if, $g_1 = g_2$, the representation is called adjoint representation, and if g_2 is equal to identity operator, the representation is called left representation of group. We will say that set of vectors $b_g = T(g)b_0$ form a family of coherent states if there exists a continuous non-zero linear functional $l_0 \in \mathcal{B}^*$ (called test functional) and a vector $b_0 \in \mathcal{B}$ (called vacuum vector) such that

$$C(b_0, b_0') = \int_G \langle T(g^{-1})b_0, l_0\rangle\langle T(g)b_0', l_0'\rangle d\mu(g), \tag{2.2}$$

is non-zero and finite, which is known as the admissibility relation. For unitary representation in Hilbert spaces, the condition (2.2) is known as square integrability. Thus our definition describes an analog of square integrable representation for Banach space.

The wavelet transform $\mathcal{W}$ from Banach space $\mathcal{B}$ to a space of function $F(G)$ that defined by a representation π of G on $\mathcal{B}$, a vacuum vector b_0 and a test functional l_0, is given by the following formula:

$$\mathcal{W} : \mathcal{B} \to F(G) : O \to \hat{O}(g) = [\mathcal{W}\hat{O}](g) = \langle T(g^{-1})\hat{O}, l_0\rangle = \langle \hat{O}, \pi^*(g)l_0\rangle \tag{2.3}$$

and the inverse wavelet transform $\mathcal{M}$ from $F(G)$ to $\mathcal{B}$ is given by the formula:

$$\mathcal{M} : F(G) \to \mathcal{B} : \hat{O}(g) \to \mathcal{M}[\hat{O}] = \int_G \hat{O}(g)b_x d\mu(g) = \int_G \hat{O}(g)T(g)b_0 d\mu(g). \tag{2.4}$$

The operator $\mathcal{P} = \mathcal{M}\mathcal{W} : \mathcal{B} \longmapsto \mathcal{B}$ is a projection of $\mathcal{B}$ into its linear subspace, in which b_0 is cyclic (i.e. the set $\{T(g)b_0 \mid g \in G\}$ span Banach space $\mathcal{B}$), and $\mathcal{M}\mathcal{W}(\hat{O}) = P(\hat{O})$ where the constant P is equal to $\frac{c(b_0,b_0')}{\langle b_0,l_0'\rangle}$. Especially, if in cases, left representation U is an irreducible representation; then, the inverse wavelet transform $\mathcal{M}$ is a left inverse operator of wavelet transform $\mathcal{W}$ on $\mathcal{B}$, i.e, $\mathcal{M}\mathcal{W} = I$.

Frames can be seen as a generalization of basis in Hilbert or Banach space. Banach frames and atomic decomposition are sequences that have basis-like properties but which need not to be bases. Atomic decomposition has played a key role in the recent development of wavelet theory.

Now we define a decomposition of a Banach space as follows:

Definition of atomic decomposition Let $\mathcal{B}$ be a Banach space, and $\mathcal{B}_d$ be an associated Banach space of scalar-valued sequences indexed by $N = \{1, 2, 3, \ldots\}$, and let $\{y_i\}_{i\in N} \subset \mathcal{B}^*$ and $\{x_i\}_{i\in N} \subset \mathcal{B}$ be given. If

(a) $\{\langle \hat{O}, y_i\rangle\} \in \mathcal{B}_d$ for each $\hat{O} \in \mathcal{B}$,
(b) The norms $\|\hat{O}\|_{\mathcal{B}}$ and $\|\{\langle \hat{O}, y_i\rangle\}\|_{\mathcal{B}_d}$ are equivalent,
(c) $\hat{O} = \sum_{i=1}^{\infty}\langle \hat{O}, y_i\rangle x_i$ for each $\hat{O} \in \mathcal{B}$,

then $(\{y_i\}, \{x_i\})$ is an atomic decomposition of $\mathcal{B}$ with respect to $\mathcal{B}_d$. In cases, the norm equivalence is given by

$$A\|\hat{O}\|_{\mathcal{B}} \le \|\{\langle \hat{O}, y_i\rangle\}\|_{\mathcal{B}_d} \le \mathcal{B}\|\hat{O}\|_{\mathcal{B}}, \tag{2.5}$$

then A, B are a choice of atomic bounds for $(\{y_i\}, \{x_i\})$.

Definition of Banach frame Let X be a Banach space, and $\mathcal{B}_d$ be an associated Banach space of scalar-valued sequences indexed by $\mathcal{N}$ and let $\{y_i\}_{i\in N} \subset \mathcal{B}^*$ and $S : \mathcal{B}_d \longrightarrow \mathcal{B}$ be given. If

(a) $\{\langle \hat{O}, y_i\rangle\} \in \mathcal{B}_d$ for each $x \in \mathcal{B}$,
(b) The norms $\|x\|_{\mathcal{B}}$ and $\|\{\langle \hat{O}, y_i\rangle\}\|_{\mathcal{B}_d}$ are equivalent,
(c) S is bounded and linear, and $S\{\langle \hat{O}, y_i\rangle\} = \hat{O}$ for each $O \in \mathcal{B}$,

then $(\{y_i\}, S)$ is a Banach frame for $\mathcal{B}$ with respect to $\mathcal{B}_d$. The mapping S is called the reconstruction operator. If the norm equivalence is given by $A\|\hat{O}\|_{\mathcal{B}} \leq \|\{\langle \hat{O}, y_i\rangle\}\|_{\mathcal{B}_d} \leq B\|\hat{O}\|_{\mathcal{B}}$, then A, B will be a choice of frame bounds for $(\{y_i\}, S)$.

Obviously one can show that the admissibility condition is the same as frame condition.

3 Quantum Tomography via Group Theory with Wavelet Transform on Banach Space

Group tomography of a compact group G, with an irreducible unitary representation U acting on separable Hilbert space $\mathcal{H}$, means that, every element of $\mathcal{B}(\mathcal{H})$, the Banach algebra of bounded linear operators acting on $\mathcal{H}$, can be constructed by the set $\{U(g), g \in G\}$ according to formula (3.2), where the set $\{U(g), g \in G\}$ is known as tomographic set and $\mathrm{Tr}[U^\dagger(g)\hat{O}]$ is sampling set or tomogram set of a given operator $\hat{O}$ [23]. When $\mathcal{H}$ is finite-dimensional, the hypothesis that $\{U(g)\}$ is a tomographic set is sufficient to reconstruct any given operator from the tomographic set by using (3.2), but the case of $\dim(\mathcal{H}) = \infty$ needs a further condition to make sure that every expression converges and that it can be attributed to a precise mathematical meaning. More explicitly, U needs to fulfill the following inequality:

$$\int d\mu(g)\langle f_1, U(g)f_2\rangle\langle f_3U(g), f_4\rangle = \langle f_1, f_4\rangle\langle f_3, f_2\rangle \quad \forall\,|f_1\langle, |f_2\rangle, |f_3\rangle, |f_4\rangle \in \mathcal{H}, \tag{3.1}$$

which is known as the square integrability of the representation $U(g)$. If O is a trace-class operator on $\mathcal{H}$ and $\{U(g)\}$ is a tomographic set and satisfies (3.1) then we have

$$\hat{O} = \int d\mu(g)\,\mathrm{Tr}[U^\dagger(g)\hat{O}]U(g). \tag{3.2}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space. In order to do so, we need choose the tomographic set $U(g)$ as a continuous representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\mathcal{W} : \mathcal{B} \mapsto F(g) : \hat{O} \mapsto \hat{O}(g) = \langle \hat{O}, l_g\rangle = \langle \hat{O}, U(g)l_0\rangle = \langle \hat{O}U(g)^\dagger, l_0\rangle = \mathrm{tr}(\hat{O}U(g)^\dagger). \tag{3.3}$$

With those conditions, the inverse wavelet transform $\mathcal{M}$ becomes a left inverse operator of the wavelet transform $\mathcal{W}$:

$$\begin{aligned}\mathcal{M}\mathcal{W} = I \Rightarrow \mathcal{M} : F(g) \mapsto \mathcal{B} : \hat{O}(g) \mapsto \mathcal{M}[\hat{O}] = \mathcal{M}\mathcal{W}(\hat{O}) = \hat{O}\\ = \int d\mu(g)\langle \hat{O}, l_g\rangle b_g.\end{aligned} \tag{3.4}$$

Therefore, with the choice of $b_0 = I$ (identity operator), the tomography relation can be written as:

$$\hat{O} = \int d\mu(g)\,\mathrm{Tr}(\hat{O}U(g)^{\dagger})U(g). \tag{3.5}$$

By choosing $b_0 = I$, $b_0' = |f_2\rangle\langle f_1|$, $l_0(\hat{O}) = \mathrm{Tr}(\hat{O}|f_4\rangle\langle f_3|)$ and $l_0'(\hat{O}) = \mathrm{Tr}(\hat{O})$ (with f_1, f_2, f_3, f_4 as arbitrary vectors in $\mathcal{H}$ and A as an arbitrary bounded operator), the admissibility condition for wavelet transform on Banach space (2.2) can be reduced to the square integrability for group theory tomography (3.1).

Similarly, by the same choice as above for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. To do so we need further to choose set $\{U(g)l_0\} \subset \mathcal{B}^*$ as the index sequence of functionals (with index set G) which belong to dual Banach space, then we can show that:

(a) $\{\langle \hat{O}, U(g)l_0\rangle\} = \{\mathrm{Tr}(\hat{O}U^{\dagger}(g))\} \in \mathcal{B}_d$ for each $\hat{O} \in \mathcal{B}$,

(b) The norms $\|\hat{O}\|_{\mathcal{B}}$ and $\|\{\mathrm{Tr}(\hat{O}U^{\dagger}(g))\}\| = [\int \mathrm{Tr}[\hat{O}U^{\dagger}(g)]\overline{\mathrm{Tr}[\hat{O}U^{\dagger}(g)]}d\mu(g)]^{\frac{1}{2}}$ are equivalent in the sense that they satisfy the inequality (2.5) with the atomic bounds $A = B = 1$, provided that we use the Hilbert–Schmidt norm for the operator $\hat{O}$ and finally by using the relation (3.2) we obtain

(c) $\hat{O} = \int \mathrm{Tr}(\hat{O}U^{\dagger}(g))U(g)d\mu(g)$.

Therefore, $\{U(g)b_0, U(g)l_0\}$ is an atomic decomposition of Banach space of bounded operators acting on representation space with respect to $\mathcal{B}_d$ with atomic bounds $A = B = 1$.

At the end by the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case, we can show that the required conditions (a), (b) for the existence of Banach frame is the same as the atomic decomposition one, and in order to have the last condition for the existence of atomic decomposition, we can define the reconstruction operator S as follows:

(c) $S\{\mathrm{Tr}(\hat{O}U^{\dagger}(g))\} = \int \mathrm{Tr}(\hat{O}U^{\dagger}(g))U(g)d\mu(g) = \hat{O}$ for each $\hat{O} \in \mathcal{B}$.

It is straightforward to show that the operator S as defined above is a linear bounded operator.

Therefore $\{U(g)l_0, S\}$ is a Banach frame for $\mathcal{B}$ with respect to $\mathcal{B}_d$ with frame bounds $A = B = 1$.

3.1 Tomography for Rotation Group

The most general (unnormalized) density matrix $\hat{\rho}$ is a $(2s+1) \times (2s+1)$ hermitian matrix with $(2s+1)^2$ real parameters. Various methods have been proposed to determine $\hat{\rho}$. The expectations of $(2s+1)^2 - 1$ linearly independent spin multipoles do fix a unique (normalized) density operator [24]. In order to establish a down-to-earth approach, it is natural to restrict the measurements to those performed with a standard Stern–Gerlach apparatus, the quantization axis of which can be arbitrarily oriented in space. Therefore, we are dealing here with Hilbert space spin tomography i.e., angular momentum tomography [11, 12, 14, 17] with Hilbert space $H = C^{2s+1}$, **s** being the spin of the particle, and the corresponding group being $SU(2)$.

Obviously, the diagonal elements of the density matrix are non-negative and their sum is equal to unity, their physical meaning being the probabilities of observing the value of spin projection on fixed axis in space. Obviously the relevant representation of $SU(2)$ group is irreducible unitary representation $D(\Omega) = D(\alpha, \beta, \gamma)$, where α, β, γ are Euler angles which are also our tomographic set at the same time. The Haar invariant measure for $SU(2)$

is given by [25]:

$$dg(\alpha,\beta,\gamma)=\frac{2s+1}{4\pi^2}\sin(\beta)d\alpha d\beta d\gamma, \tag{3.6}$$

and finally spin tomography can be written as:

$$\rho=\frac{2s+1}{4\pi^2}\int_0^{2\pi}\frac{2s+1}{4\pi^2}\sin(\beta)d\alpha d\beta d\gamma\,\mathrm{Tr}[\rho D^\dagger(\alpha,\beta,\gamma)]D(\alpha,\beta,\gamma). \tag{3.7}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space. In order to do so, we need choose the tomographic set $D(\alpha,\beta,\gamma)$ as a continuous representation of the wavelet transformation and the identity operator as a vacuum vector. Therefore, the corresponding wavelet transformation takes the following form:

$$\begin{aligned}&\mathcal{W}:\mathcal{B}\mapsto F(G):\hat{\rho}\mapsto\hat{\rho}(\alpha,\beta,\gamma)\\&\quad=\langle\hat{\rho},l_{(\alpha,\beta,\gamma)}\rangle=\langle\hat{\rho},D(\alpha,\beta,\gamma)l_0\rangle=\langle\hat{\rho}D^\dagger(\alpha,\beta,\gamma),l_0\rangle=\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma)).\end{aligned} \tag{3.8}$$

With this condition, inverse wavelet transform in $\mathcal{M}$ is a left inverse operator on B for the wavelet transform $\mathcal{W}$:

$$\begin{aligned}&\mathcal{M}:F(G)\mapsto\mathcal{B}:\hat{\rho}(\alpha,\beta,\gamma))\mapsto\mathcal{M}[\hat{\rho}]=\mathcal{M}(\hat{\rho}),\\&\mathcal{M}\mathcal{W}(\hat{\rho})=\int d\mu(\alpha,\beta,\gamma)\langle\hat{\rho},l_{(\alpha,\beta,\gamma)}\rangle b_{(\alpha,\beta,\gamma)}.\end{aligned} \tag{3.9}$$

Also the constant on left hand side of (2.2) becomes proportional to the dimension of unitary representation, that is, $C(b_0,b'_0)=2s+1=d$, where **d** is dimensional of representation. Finally the constant P becomes equal to one, i.e., $P=\frac{c(b_0,b'_0)}{\langle b_0,l'_0\rangle}=1$, and the reconstruction procedure of wavelet transform (operation of the combination of wavelet transform and its inverse one, $\mathcal{M}\mathcal{W}$ on the density operator $\hat{\rho}$) leads to the tomography relation (3.7).

Again, by the same choice as above for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. To do it, we need just choose the set $\{D(\alpha,\beta,\gamma)l_0\}\subset\mathcal{B}^*$ as the index sequence of functionals (with index set G) which belong to dual Banach space, then we can show that:

(a) $\{\langle\hat{\rho},D(\alpha,\beta,\gamma)l_0\rangle\}=\{\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))\}\in\mathcal{B}_d$ for each $\hat{\rho}\in\mathcal{B}$,

(b) The norms $\|\hat{\rho}\|_{\mathcal{B}}$ and $\|\{\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))\}\|$ are equivalent in the sense that they satisfy the inequality (2.5) with the atomic bounds $A=B=1$, provided that we use the Hilbert–Schmidt norm for the operator $\hat{\rho}$:

$$\begin{aligned}&\int\|\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))\|d\mu(\alpha,\beta,\gamma)\\&\quad=\int\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))\overline{\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))}d\mu(\alpha,\beta,\gamma)\\&\quad=\sum_{m,m',n,n'}\int\hat{\rho}^J_{mm'}\hat{\rho}^{*J}_{nn'}D^J_{nn'}(\alpha,\beta,\gamma)D^{*J}_{mm'}(\alpha,\beta,\gamma)d\mu(\alpha,\beta,\gamma)=\|\hat{\rho}\|^2\end{aligned} \tag{3.10}$$

and if we use the relation (3.7), we have:

(c) $\hat{\rho}=\int\mathrm{Tr}(\hat{\rho}D^\dagger(\alpha,\beta,\gamma))D(\alpha,\beta,\gamma)d\mu(\alpha,\beta,\gamma)$.

Therefore, $\{D(\alpha,\beta,\gamma)b_0, D(\alpha,\beta,\gamma)l_0\}$ is an atomic decomposition of Banach space of bounded operators acting on spin **s** representation space with respect to $\mathcal{B}_d$ with atomic bounds $A = B = 1$.

Finally, the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case yield the required conditions (a) and (b) for the existence of Banach frame, which is the same as the atomic decomposition one, and in order to have the last condition for the existence of atomic decomposition, we define the reconstruction operator S as follows:

(c) $S\{\mathrm{Tr}(\hat{\rho}D^{\dagger}(\alpha,\beta,\gamma))\} = \int \mathrm{Tr}(\hat{\rho}D^{\dagger}(\alpha,\beta,\gamma))D(\alpha,\beta,\gamma)d\mu(\alpha,\beta,\gamma) = \hat{\rho}$ for each $\hat{\rho} \in \mathcal{B}$.

Also it is straightforward to show that the operator S as defined above is a linear bounded operator.

Therefore, $\{D(\alpha,\beta,\gamma)l_0, S\}$ is a Banach frame for Banach space of operators acting on spin representation space with respect to $\mathcal{B}_d$ with frame bounds $A = B = 1$.

Let us first introduce $(2s+1)^2$ irreducible multipole tensor operators T_{LM} [26]:

$$\widehat{T}^{s}_{LM} = \sqrt{\frac{2L+1}{2S+1}} \sum_{m,m'=-s}^{s} |sm\rangle\langle sm'| C^{sm}_{sm'LM}. \tag{3.11}$$

By substituting the following identity:

$$\widehat{D}^{s}(\alpha,\beta,\gamma) = \sum_{L,M,m,m'} \sqrt{\frac{2l+1}{2s+1}} C^{sm}_{sm'LM} D^{s}{}_{mm'}(\Omega)\widehat{T}^{s}_{LM}. \tag{3.12}$$

in (3.7) where $C^{sm}_{sm'LM}$ are Clebsch–Gordan coefficients, and by using the sums involving product of two Clebsch–Gordan coefficients, we get:

$$\hat{\rho} = \sum_{LM} \mathrm{Tr}[\hat{\rho}\widehat{T}^{s\dagger}_{LM}]\widehat{T}^{s}_{LM}. \tag{3.13}$$

On the other hand, we know that these operators are the most convenient basis for $\mathcal{B}$, Banach space of operators acting on $(2s+1)$-dimensional Hilbert space associated with the representation of rotation group, in sense that any linear bounded operator acting in the $(2s+1)$-dimensional spin state space can be expanded in terms of these multipole tensor operators [13].

3.2 Tomography of Quantum Spinor States

The family of the probability distribution functions of the $1/2$-spin projection is parameterized by the point's coordinates θ and φ on the sphere of unity radius. This parameterization coincides with the physical meaning of the marginal distribution in the sense that the distribution function $w(m,\Omega)$ is the probability to observe the spin projection m if we measure this spin projection on the quantization axis which is parallel to the vector normal to the surface of the sphere of the unit radius at the point with the coordinates θ and φ. If we know the positive and normalized marginal distribution $w(m,\Omega)$, then, as it was shown in [15, 16, 27], the matrix elements $\hat{\rho}^{(j)}_{ik}$ can be calculated with the help of the measurable marginal distribution $w(m,\Omega)$ of the particle with an arbitrary spin j and the values of indices

$m = -j, -j+1, \ldots, j$ by means of the relation:

$$(-1)^k \hat{\rho}_{ik}^{(j)} = \sum_{J'=0}^{2j} \sum_{M=-J'}^{J'} (2J'+1)^2 \sum_{m=-j}^{j} (-1)^m \otimes \int w(m,\Omega) D_{0M}^{J'}(\Omega) \frac{d\Omega}{8\pi^2} \begin{pmatrix} j & j & J' \\ m & -m & 0 \end{pmatrix} \begin{pmatrix} j & j & J' \\ i & -k & M \end{pmatrix}, \quad (3.14)$$

where $i, k = -j, -j+1, \ldots, j$ and the integration has been operationalized according to the rotation angles φ, θ, ψ

$$\int d\Omega = \int_0^{2\pi} d\varphi \int_0^{2\pi} d\psi \int_0^{\pi} \sin\theta d\theta. \quad (3.15)$$

Since in this example adjoint representation is used, the ψ parameter is removed and the representation is limited to homogeneous space, but the representation which we used was group representation. Therefore, the spin tomography can be obtained via wavelet transform in Banach space on Homogeneous space $S^2 = SU(2)/U(1)$, for more details see [28].

Then definition of the wavelet transforms for adjoint representation is given by:

$$\hat{\rho}(\Omega) = \langle \hat{\rho}, l_\Omega \rangle = \langle T(\Omega)\hat{\rho}, l_0 \rangle = \langle U(\Omega)\hat{\rho}(\Omega)^\dagger, l_0 \rangle, \quad (3.16)$$

where U is irreducible representation of $SU(2)$ group for spin J. In this case, by choosing the test functional as $l_0(\hat{\rho}) = \mathrm{Tr}[\hat{\rho}|j,m_1\rangle\langle j,m_2|]$, the corresponding wavelet transform becomes:

$$\begin{aligned} \hat{\rho}(\Omega) &= \langle U(\Omega)|j,m_1\rangle\langle j,m_2|U(\Omega)^\dagger, \hat{\rho}\rangle \\ &= \langle j,m_1|U(\Omega)\hat{\rho}U(\Omega)^\dagger|j,m_2\rangle = \omega(\Omega,m_1,m_2). \end{aligned} \quad (3.17)$$

The inverse wavelet transform is:

$$\mathcal{M}(\hat{\rho}) = \int d\Omega\, \omega(\Omega,m_1,m_2) U^\dagger(\Omega) b_0 U(\Omega), \quad (3.18)$$

if we choose $b_0 = |j,m_1\rangle\langle j,m_2|$, the inverse wavelet transform becomes:

$$\mathcal{M}(\hat{\rho}) = \int d\Omega\, \omega(\Omega,m_1,m_2) U^\dagger(\Omega)|j,m_1\rangle\langle j,m_2|U(\Omega). \quad (3.19)$$

Therefore, the matrix elements $\mathcal{M}(\hat{\rho})_{ik}$ take the following form:

$$\mathcal{M}(\hat{\rho})_{ik} = \int d\Omega\, \omega(\Omega,m_1,m_2)\langle j,i|U^\dagger(\Omega)|j,m_1\rangle\langle j,m_2|U(\Omega)|j,k\rangle. \quad (3.20)$$

By putting $m_1 = m_2 = m$, and taking the average sum over m, we get:

$$\mathcal{M}(\hat{\rho})_{ik} = \left(\frac{1}{2j+1}\right) \sum_m \int d\Omega\, \omega(\Omega,m)(-1)^{m-j} D_{im}^j(\Omega) D_{-k-m}^j(\Omega), \quad (3.21)$$

now, if we expand the product of D functions in the terms of D function (by using addition rule of D functions), after some algebra we get:

$$\mathcal{MW}\hat{\rho}_{ik}^{(j)} = \left(\frac{1}{2j+1}\right)\sum_{J'=0}^{2j}\sum_{M=-J'}^{J'}(2J'+1)^2\sum_{m=-j}^{j}(-1)^{m-k} \otimes \int w(m,\Omega)D_{0M}^{J'}(\Omega)\frac{d\Omega}{8\pi^2}\begin{pmatrix} j & j & J' \\ m & -m & 0 \end{pmatrix}\begin{pmatrix} j & j & J' \\ i & -k & M \end{pmatrix}, \quad (3.22)$$

also after some routine calculation, we can show that the constant on left hand side of (2.2) is $C(b_0, b_0') = 2j+1$, the constant $P = \frac{c(b_0,b_0')}{\langle b_0, l_0' \rangle}$ is equal to $\frac{1}{2j+1}$, and finally the reconstruction procedure of wavelet transform (operation of the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the density operator $\hat{\rho}$) leads to the tomography relation (3.14).

Again, by the above choice of vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example. To do it, we need just choose the set $\{T(\Omega)l_0\} \subset \mathcal{B}^*$ as the index sequence of functionals (with index set G) which belong to dual Banach space, then we can show the following conditions:

(a) $\{\langle \hat{\rho}, T(\Omega)l_0 \rangle\} = \{\mathrm{Tr}(T^\dagger(\Omega)\hat{\rho})\} \in \mathcal{B}_d$ for each $\hat{\rho} \in \mathcal{B}$,

(b) The norms $\|\hat{\rho}\|_{\mathcal{B}}$ and $\|\{\mathrm{Tr}(U(\Omega)\hat{\rho}U^\dagger(\Omega))\}\|$ are equivalent in the sense that they satisfy the inequality (2.5) with the atomic bounds $A = B = \frac{1}{2j+1}$, providing that we use the Hilbert–Schmidt norm for the density operator $\hat{\rho}$:

$$\int \|\mathrm{Tr}(U^\dagger(\Omega)\hat{\rho}U(\Omega))\| d\mu(\Omega) = \int \mathrm{Tr}(U(\Omega^{-1})\hat{\rho}U(\Omega))\overline{\mathrm{Tr}(U(\Omega^{-1})\hat{\rho}U(\Omega))} d\mu(\Omega), \quad (3.23)$$

if we use the relation (3.22), we have:

(c) $\hat{\rho} = \int \mathrm{Tr}(U(\Omega^{-1})\hat{\rho}U(\Omega))U(\Omega)b_0U(\Omega^{-1})d\mu(\Omega)$.

Therefore, $\{T(\Omega)b_0, T(\Omega)l_0\}$ is an atomic decomposition of Banach space of bounded operators acting on spin **s** representation space with respect to $\mathcal{B}_d$ with atomic bounds $A = B = \frac{1}{2j+1}$.

Finally, the same choice of vacuum vector, test functional and index sequence of functional as in the atomic decomposition case yield the required conditions (a) and (b) for the existence of Banach frame which is the same as the atomic decomposition one, and in order to have the last condition for the existence of atomic decomposition, we define the reconstruction operator S as follows:

(c) $S\{\mathrm{Tr}(U(\Omega^{-1})\hat{\rho}U(\Omega))\} = \int \mathrm{Tr}(U(\Omega^{-1})\hat{\rho}U(\Omega))U(\Omega)b_0U(\Omega^{-1})d\mu(\Omega) = \hat{\rho}$ for each $A \in \mathcal{B}$.

Also it is straightforward to show that the operator S as defined above is a linear bounded operator.

Therefore, $\{T(\Omega)l_0, S\}$ is a Banach frame for Banach space of operators acting on spin representation space with respect to $\mathcal{B}_d$ with frame bounds $A = B = \frac{1}{2j+1}$.

3.3 Discrete Spin Tomography

It is easy to characterize a class of discrete representations whose reconstruction is similar to the continuous representation. The most general form of a density matrix of a single qubit can be written as:

$$\rho(\vec{n}) = \frac{1}{2}(1 + a\vec{n}.\vec{\sigma}),$$

where $0 \le a \le 1$ and the unit vector in Bloch sphere $\vec{n}$ is the direction along which the spin is pointing up, and it satisfies the following conditions [29]:

$$\sum_{\alpha=1}^{K} \vec{n}_\alpha = 0, \qquad \frac{1}{3}\delta_{jk} = \frac{1}{K}\sum_{\alpha}(n_\alpha)_j(n_\alpha)_k. \tag{3.24}$$

In the dihedral and tetrahedral subgroups of $SU(2)$, these condition are satisfied by unit vectors that point to the vertices of any regular polyhedron inscribed within the Bloch sphere i.e., a tetrahedron, octahedron, cube, icosahedron, or dodecahedron. The four projectors for a tetrahedron are linearly independent. An octahedron gives rise to the six cardinal-direction representations on the Bloch sphere. The regular polyhedral by no means exhausts the possibility of representations of this sort. we can use simultaneously the vertices forms of any number of polyhedral.

For spin $s = 1$ it is possible to find a finite group instead of $SU(2)$. In fact, consider the 12 element Tetrahedric group composed of the $\pm\frac{2\pi}{3}$ rotations around the versos:

$$\left\{\vec{n}_1 = \frac{1}{\sqrt{3}}(1,1,1), \vec{n}_2 = \frac{1}{\sqrt{3}}(1,-1,-1), \vec{n}_3 = \frac{1}{\sqrt{3}}(-1,1,-1), \vec{n}_4 = \frac{1}{\sqrt{3}}(-1,-1,1)\right\},$$

of the π rotations around

$$\{\vec{n}_5 = (1,0,0), \vec{n}_6 = (0,1,0), \vec{n}_7 = (0,0,1)\},$$

and of the identity. It induces a unitary irreducible representation on the space C^3, given by 3×3 rotation matrices. Hence, the tomography for $S = 1$ can be written as [14, 17]:

$$\hat{\rho} = \frac{1}{4}\sum_{m=-1}^{1}\sum_{j=1}^{7} P(\vec{n}_j, m)K_j(m - \vec{s}.\vec{n}_j) + \frac{1}{4}I, \tag{3.25}$$

where $P(\vec{n}_j, m)$ is the probability of having outcome **m** which is the result of measuring the operator $\vec{s}.\vec{n}$ and $K_j(m - \vec{s}.\vec{n}_j)$ is a kernel function representation $e^{i\Psi(\vec{s}.\vec{n})}$ and the identity operator as the vacuum vector, the corresponding wavelet transform becomes:

$$\begin{aligned}\mathcal{W} &: \mathcal{B} \mapsto F(n,\Psi) : \hat{\rho} \mapsto \hat{\rho}(n,\Psi)\\ &= \langle \hat{\rho}, l_{(n,\Psi)}\rangle = \langle \hat{\rho}, R_n(\Psi)l_0\rangle = \langle R_n^\dagger(\Psi)\hat{\rho}, l_0\rangle = \mathrm{tr}(R_n^\dagger(\Psi)\hat{\rho}).\end{aligned} \tag{3.26}$$

For the choice of the identity operator as a vacuum vector and the test functional $l_0(\hat{\rho}) = \mathrm{Tr}[\hat{\rho}]$, the inverse wavelet transform $\mathcal{M}$ (2.4) becomes left inverse operator of the wavelet transform $\mathcal{W}$:

$$\begin{aligned}\mathcal{M}\mathcal{W} = PI &\Rightarrow \mathcal{M} : F(n,\Psi) \mapsto \mathcal{B} : \hat{\rho}(n,\Psi) \mapsto \mathcal{M}[\hat{\rho}] = \mathcal{M}\mathcal{W}(\hat{\rho})\\ &= \sum_{\Psi,n}\frac{1}{P}\langle \hat{\rho}, l_{(n,\Psi)}\rangle b_{(n,\Psi)}.\end{aligned} \tag{3.27}$$

Evaluating the trace over the complete set of vectors $|\vec{n}, m\rangle$, which are the eigenstate of $\vec{S}.\vec{n}$, with eigenvalues m, and by taking into account $\langle\vec{n}, m|\hat{\rho}|\vec{n}, m\rangle = p(\vec{n}, m)$, we get:

$$\mathcal{MW}(\hat{\rho}) = \frac{1}{P}\sum_{m=-1}^{1}\sum_{j=1}^{7} P(\vec{n}_j, m)K_j(m - \vec{s}.\vec{n}_j) + \frac{1}{P}I \tag{3.28}$$

we can show that the constant, on left hand side of (2.2) $C(b_0, b_0') = 12$, the constant $P = \frac{c(b_0,b_0')}{\langle b_0,l_0'\rangle} = 4$, and finally the reconstruction procedure of wavelet transform (operating the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the operator $\hat{\rho}$) leads to the tomography relation (3.25).

Again, by the above choice for vacuum vectors and test functions, we can get the atomic decomposition and Banach frame for this example similar to rotation group with bound $A = B = 4$.

3.4 Unitary Group Tomography

Here in this case we consider a state with d levels. Firstly we prepare the generators for $SU(d)$ systems and thereby construct the density matrices for a qudit system. By choosing an irreducible square integrable representation of $SU(d)$ group as $U(\Omega) = e^{i\hat{J}.\hat{n}\psi}$, the tomography relation is given by:

$$\rho = \int d\mu(\Omega)\,\mathrm{Tr}[U^\dagger(\Omega)\rho]U(\Omega). \tag{3.29}$$

Now we try to obtain the above explained tomography via wavelets transform in Banach space. The wavelet transform $\mathcal{W}$ from Banach space $\mathcal{B}$ to a space of function $F(G)$ is defined by a representation $U(\Omega) = e^{i\hat{J}.\hat{n}\psi}$ of G on $\mathcal{B}$, with the selection of a vacuum vector b_0 which is equal to identity, and a test functional $l_0(\hat{\rho}) = \mathrm{Tr}(\hat{\rho})$ is given by the following formula:

$$\mathcal{W} : \mathcal{B} \longmapsto F(\Omega) : \rho \longmapsto \hat{\rho}(\Omega) = \langle\rho, l_\Omega\rangle = \langle U^\dagger(\Omega)\rho, l_0\rangle = \mathrm{Tr}[U^\dagger(\Omega)\rho]. \tag{3.30}$$

Also the constant on left hand side of (2.2) becomes proportional to the dimension of the unitary representation, that is, $C(b_0, b_0') = d$, where d is dimensional of representation. Finally the constant P becomes equal to one, i.e., $P = \frac{c(b_0,b_0')}{\langle b_0,l_0'\rangle} = 1$, and the reconstruction procedure of wavelet transform (operating the combination of wavelet transform and its inverse one, $\mathcal{MW}$ on the density operator $\hat{\rho}$) leads to the tomography relation (3.29).

We know that we can expand the $SU(d)$ representation in terms of the generators J as:

$$U(\Omega) = \sum_{i=0}^{d^2-1} a_i(\Omega)J_i, \quad a_i(\Omega) = \frac{1}{d}\mathrm{Tr}[U(\Omega)J_i], \tag{3.31}$$

and using the relation $\mathrm{Tr}[J_iJ_j] = d\delta_{ij}$, the tomography relation can be written as:

$$\rho = \int d\mu(\Omega)\sum_{i,j}\mathrm{Tr}[J_i\rho]J_j a_i^*(\Omega)a_j(\Omega). \tag{3.32}$$

Now using the identity:

$$\int d\mu(\Omega) a_i^*(\Omega) a_j(\Omega) = \frac{1}{d}\delta_{ij}, \tag{3.33}$$

the tomography relation reduces to:

$$\rho = \frac{1}{d}\sum_j \mathrm{Tr}[J_i \rho] J_i. \tag{3.34}$$

The generators of $SU(d)$ group may be conveniently constructed by the elementary matrices of d-dimensions, $\{e_j^k \mid k, j = 1, \ldots, d\}$.

$$(e_j^k)_{\mu\nu} = \delta_{\nu j}\delta_{\mu k}, \quad 1 \le \nu, \mu \le d. \tag{3.35}$$

There are $d(d-1)$ traceless matrices,

$$\Theta_j^k = e_j^k + e_k^j, \tag{3.36}$$

$$\beta_j^k = -i(e_j^k - e_k^j), \quad 1 \le k < j \le d, \tag{3.37}$$

which are the off-diagonal generators of the $SU(d)$ group. We add the $d-1$ traceless matrices

$$\eta_r^r = \sqrt{\frac{2}{r(r+1)}}\left[\sum_{j=1}^{r} e_j^j - r e_{r+1}^{r+1}\right], \tag{3.38}$$

as the diagonal generators and obtain a total of $d^2 - 1$ generators. $SU(2)$ generators are, for instance, given as $\{X = \Theta_2^1 = e_2^1 + e_1^2, Y = \beta_2^1 = -i(e_2^1 - e_1^2), Z = \eta_1^1 = e_1^1 - e_2^2\}$.

We now define the λ-matrices, which are similar to Pauli matrices in $SU(2)$ case:

$$\lambda_{(j-1)^2+2(k-1)} = \Theta_j^k, \tag{3.39}$$

$$\lambda_{(j-1)^2+2k-1} = \beta_j^k, \tag{3.40}$$

$$\lambda_{j^2-1} = \eta_{j-1}^{j-1}. \tag{3.41}$$

In conjunction with a scaled d-dimensional identity operator these form a complete hermitian operator basis [18]. If we replace J_i with $\lambda_i/2$, unitary group tomography relation (3.34), reduced to

$$\hat{\rho}_d = \frac{1}{d}\sum_{j=0}^{d^2-1} r_j \hat{\lambda}_j. \tag{3.42}$$

In which $\widehat{\rho_d}$ is a density matrix of dimension d, a qudit, and $\mathrm{Tr}[\widehat{\rho_d}] = 1$ implies that the coefficient r_0 is one. The condition $\mathrm{Tr}[\rho_d^2] \le 1$ requires $\sum_{j=1}^{d^2-1} r_j^2 \le d(d-1)/2$.

At the end we can show that $\{U(\Omega)b_0, U(\Omega)l_0\}$ is atomic decomposition and $(\{U(\Omega)l_0\}, S)$ is Banach frame with atomic bounds $A = B = 1$.

We can extend these results to n-qudits. It is shown that for multiple qudits one needs only to consider a space of operators defined by the tensor product of the generators, $SU(d) \otimes SU(d) \otimes \cdots \otimes SU(d)$, and the representation is the tensor product of the representations, where this representation is irreducible. If we choose the vacuum vector as $b_0 = I_d \otimes I_d \otimes$

$\cdots \otimes I_d$ for admissibility condition we will have $C(b_0, b_0') = d^n$. Then n-qudit tomography can be written as:

$$\hat{\rho}_{nd} = \frac{1}{d^n} \sum_{j1\ldots jn=0}^{d^2-1} r_{j1\ldots jn} \hat{\lambda}_{j1} \otimes \cdots \otimes \hat{\lambda}_{jn}, \tag{3.43}$$

where, $r_{00\ldots0} = 1$ and $r_{j1,j2,\ldots,jn} = \frac{d^n}{2^n} \mathrm{Tr}[\lambda_{j1} \otimes \lambda_{j2} \ldots \otimes \lambda_{jn} \rho]$ $(j1, j2, \ldots, jn = 1, 2, \ldots, d)$.

4 Conclusions

The Banach space wavelets transformation nature of quantum tomography of mixed quantum states has been revealed. Also by considering various well known examples of quantum tomography, it is shown that, the quantum tomography of mixed quantum states, are almost the same as the Banach space wavelets reconstruction formalism associated with some unitary representation of finite or infinite group. Using this fact, the frame and atomic decomposition nature of quantum tomography of mixed states is also explained.

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